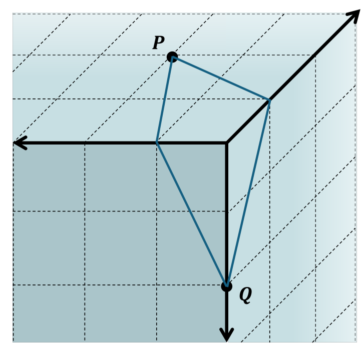
DIVERGING MATHEMATICS

Sample Adventures in Non-Euclidean Geometry

The year starts with the expedition described on this page. Additional adventures, like the ones outlined on the second page, will be organized as time permits and interest dictates, or perhaps become the foundation of future seminars on this topic.

Breaking out of the confines of Euclid's five postulates is remarkably simple to describe. Suppose we start with the Euclidean plane, remove the 4th quadrant, and then glue the negative y-axis to the positive x-axis (so that the point $(0, -a)$ is identified with the point $(a, 0)$).

This process only describes what to do if you wander around the plane, so it makes sense to think of it as *intrinsic* to the surface. If you do this with a piece of paper, some scissors and a piece of tape, the page will pop out at the origin, and with a little folding you will see the corner of a cube. Because this required us to use the ambient space around the plane, we call this an *extrinsic* description of the surface. Now, consider these two points on that surface shown in the image on the right.



Unfolding the cube, convince yourself that both line segments $\overline{PQ}$ are straight. It is important to resist the urge to use the extrinsic picture and imagine a path from P to Q that travels *through* the cube. Remember - that's not part of our space and would be nonsense inside the intrinsic definition we started with.

Consider the following questions:

1. What is the length of the shortest path from P to Q? Start with the ones that are drawn, then try other candidates you think could be shorter.

2. Find the angles formed at P and Q.

3. Choose a different point on the 'top' for P and find the new side lengths and angles. Note that the side lengths may be different but must be straight when unfolded. The shortest path is called a *geodesic*, while longer paths are said to be *locally geodesic* if they minimize distance when you zoom in (so they appear 'straight').

4. Suppose you moved both P and Q. Will they always form a bigon? Why / Why not?

5. Consider the interior angle sum for the bigons you have seen. Is there a pattern?

We will start our first meeting by discussing these questions and sharing discoveries and questions that you encountered along the way.

Additional Adventures:

Here is an incomplete list of ideas for additional expeditions in geometry that flout Euclid's tyranny.

1. The expedition on the cube, outlined on the previous page, is a special case of the geometry on an infinite cone. Suppose that instead of a quadrant, the space between two rays emanating from the origin separated by angle θ is removed and the rays are identified. How much of Euclid's *Elements* hold up to this little tweak? Are there updates we can make to definitions and postulates that lead to similar theorems?

2. What if there are multiple cone points? This happens on the cube (or any other polyhedron). Is it possible to discover the shape of our space *intrinsically*? What insight can we get about how polyhedra are formed and what it would mean for them to enclose a volume?

3. What other shapes can we make out of flat materials besides polyhedra? Cylinders and cones are a great place to start, as the symmetry can be exploited to dig into what's happening along the smooth seams. The jump to sports balls is a fun one, as are applications in industrial design and medicine.

4. We have seen how to detect seams and points, but what about holes? How would an ant wandering around a surface know the bigger distinctions like between the plane, infinite cylinder, or the donut (torus)? These questions are an entry point into Topology, one of the 'big' branches in modern mathematics.

5. Another way to poke holes in Euclid's geometry is through Fractals like the Sierpiński Carpet (pun intended). How much of a surface can you remove before it's a line? Can lines ever fill up an entire surface? Questions like this led to a fundamental breakdown of mathematician's confidence in the 1800s, jeopardizing their perch as the Queen of the Sciences. Don't worry, order was restored.

6. We have seen curvature concentrated and points and along seams, but what can we say if it is spread over a surface? The intuition developed could lead us to explore the surface of the sphere and the hyperbolic plane. Spherical Trigonometry was once a critical part of educating mariners and astronomers, but with the advent of the computer this topic has all but been eliminated from the curriculum.