Sample Adventures in Contemplating Infinity with Algebra

The year starts with the expedition described on this page. Additional adventures, like the ones outlined on the second page, will be organized as time permits and interest dictates, or perhaps become the foundation of future seminars on this topic.

Do you remember the early days of Algebra, when the most complicated thing you saw was a polynomial? Now you have to deal with things like this:

$$f(x) = \frac{x^3 - 8x^2 - 6x + 10}{x - 10}$$

If you pine for those simpler days, then you may be interested to know that this function is *almost* a polynomial. In fact, lots of functions are! The key is to narrow your focus a bit.

In this adventure, we will zoom way out, and look at this function from 10,000 ft and notice that it's a perfectly nice parabola. We can then zoom in and the origin under a microscope, and that too is a nice parabola. Naturally, these are not the same parabola - life is never that simple!

1. Find a quadratic function $g(x)$ so that

   $$f(x) = g(x) + \frac{A}{x - 10},$$

   where A is just some number.

   Plot $f(x)$ and $g(x)$ and compare them when $x \gg 1$.

2. Find a quadratic function $\ell(x)$ so that

   $$f(x) = \ell(x) + \frac{Bx^3}{x - 10},$$

   where B is just some number.

   Plot $f(x)$ and $\ell(x)$ and compare them when $x \approx 0$.

3. Choose another rational function where the numerator is a cubic and the denominator is linear.

   (a) Is the global behavior ($x \gg 1$) always quadratic?

   (b) Is the local behavior ($x \approx 0$ always quadratic?

We will start our first meeting by discussing these questions and sharing discoveries and questions that you encountered along the way.

DIVERGING MATHEMATICS

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Additional Adventures:

Here is an incomplete list of ideas for additional expeditions in algebra that will surprise and amaze you.

1. Algebra is the vehicle, but the real goal of this course is to contemplate the infinite. Zeno's Paradoxes, Hilbert's Hotel, and the Continuum Hypothesis will liven up the adventure.

2. The expedition with rational functions, outlined on the previous page, was just the tip of the iceberg. The global behavior is fairly tame, and a topic occasionally covered in Algebra courses, but the local behavior is a whole different can of worms. We were only looking for quadratic approximations, but you could have searched for approximations of *any* degree. It's only a small step before we talk about power series!

3. With the introduction of a new "thing", we will need to talk about the usual questions: Can we add, subtract, multiply (*gasp!*), divide (*gulp*), and compose (*!?!?!?*) these things? Fortunately, we have a way to verify that things are working correctly, but getting comfortable with power series will take some time. And patience.

4. Rational functions were an interesting place to start, but what about other functions we learned in algebra, like Radical functions, Exponential functions, and Logarithmic functions? There's also all those geometric functions from Trigonometry: can we do those?

5. You saw imaginary numbers when working with the quadratic formula, but they weren't very useful for getting *real* solutions. This is why mathematicians avoided them for millennia, until progress was made solving the cubic (or for the geometers out there: trisecting an angle). Finally imaginary numbers had value in the real world. Of course this is now just a footnote in the history of imaginary numbers, but also an undeniable adventure.